The geometry of double-scattering waves in 3-3 collisions

1986 J. Phys. A: Math. Gen. 19 725

(http://iopscience.iop.org/0305-4470/19/5/026)

J. Phys. A: Math. Gen. 19 (1986) 725-741. Printed in Great Britain

The geometry of double-scattering waves in 3–3 collisions†

S Servadio
Dipartimento di Fisica, Università degli Studi di Pisa, piazza Torricelli, 2, Pisa, 56100 Italy

Received 16 November 1984, in final form 17 July 1985

Abstract. The geometry associated with the double-scattering contribution to elastic 3-3 collisions is investigated in detail. This makes it possible to separate clearly the dynamical content of the leading asymptotic term contributed by on-mass-shell intermediate states and to calculate by quadratures the next-to-leading term. The relevance of the present results is discussed in connection with flux calculations.

1. Introduction

Despite its modest direct experimental relevance the elastic 3-3 scattering has a bearing on several different questions of physics. Due to unitarity, the elastic scattering is always linked to inclusive reactions of the type 3-anything, whether the scattering objects are 'elementary' particles, atoms or molecules. Its role must then be unambiguously assessed even if one is more interested in inelastic processes.

A different but related problem is the calculation of the third virial coefficient for a real fluid. Here one must assess the weights of a complete set of states and thus the relative weights of asymptotically different states.

Both the above problems have been solved within the framework of non-relativistic quantum mechanics (Faddeev 1965, Buslaev and Merkurev 1970) in the p-space representation, but the answers are difficult to find from an x-space point of view (Gerjuoy 1971) and perhaps not phrased in the most useful manner (Kirzhnits and Takibaev 1978). Only after Faddeev, in his fundamental work, clarified the p-space singularities of the elastic scattering was it possible to successfully investigate the x-space structure of the associated wavefunction (Merkurev 1971, Nuttall 1971) and ask questions about a meaningful definition of a three-body cross section (Newton and Shtockhamer 1976, Potapov and Taylor 1977a, b) and the so-called 'optical theorem' for 3-3 scattering (Servadio 1981b).

In the early stages of this work it was already evident that the study of the wavefunction should be pushed one step further than before (Servadio 1981a) in order to be able to take the study of the fluxes to the final O(1) order. In so doing, however, I became convinced that the double-scattering wavefunction had not been cast into the best shape. In the leading terms of its asymptotic expansion one could not recognise the geometry that the asymptotically free wave must embody: namely, the fact that as the wavefront progresses the intensity must decrease according to some expansion coefficient related to the radii of curvature of the wavefront. When reinterpreted along these lines, each scattering term should be analysed by separating its

† Work supported in part by INFN Sezione di Pisa.

geometrical from its dynamical (t-matrix) contents. Moreover, such a representation should be very illuminating when calculating the next-to-leading-order terms of the asymptotic expansion by imposing the asymptotically free wave equation.

Soon after, a quite general theorem was proved (Servadio 1983) stating that each scattering term is characterised, within angular sectors of uniform asymptotic behaviour, by a wavefront that is either a spherical or a $K=0$ ruled surface.

The purpose of the present paper is to corroborate explicitly all these contentions on the double-scattering wavefunction Φ^{13}. The main new result is the formula for the amplitude A_1 of the next-to-leading order as given by (5.1). This formula is, apart from elementary quadratures and cancellations to be explained in the appendix, quite explicit and gives A_1 in terms of the dynamical (t-matrix) factor $\mathcal{J}(\xi)$ and the geometry (the radii R_i) of the wavefront. This separation of dynamics from geometry is very important when the term (A_1/ρ^3) is used in O(1) flux calculations, as will be briefly commented on in § 6.

The plan of the paper is as follows. In § 2 we shall first show the full asymptotic structure of Φ^{13} into 'non-dangerous' directions in which the particles are asymptotically free. In § 3 we shall investigate the bearing of the wave equation on Φ^{13}. The eikonal equation for the wavefront profile function and Luneburg's recurrence relation for the amplitude will be the main results. The geometry of the ruled surface corresponding to the double scattering with on-shell intermediate state propagation will be fully shown in § 4. The leading amplitude will be written in geometric terms along the same lines as Keller's geometrical theory of diffraction. In § 5 the next terms will be obtained by integration of the recurrence relation and unexpected logarithmic terms will appear. Finally, the appendix proves the complete cancellation of the logarithmic terms stemming from the following geometrical fact: the ruled surface, with the parametrisation induced by its lines of principal curvature, is embedded in the larger space R_6 which is flat.

2. Double scattering to higher order

Let us consider, within the framework of non-relativistic quantum theory, the system of three particles interacting by pair potentials, initially in a state of definite momenta as viewed from their centre-of-mass system.

The masses are taken to be equal, $m_i = 1$; if $\boldsymbol{r}_i$ and $\boldsymbol{p}_i$ are the positions and momenta of the particles, relative coordinates and conjugate momenta can be chosen in a standard form (Servadio 1981a) as

$$\boldsymbol{X}_1 = \sqrt{\tfrac{3}{2}}\boldsymbol{r}_1, \qquad \boldsymbol{P}_1 = \sqrt{\tfrac{3}{2}}\boldsymbol{p}_1,$$
$$\boldsymbol{Y}_1 = (1/\sqrt{2})(\boldsymbol{r}_2 - \boldsymbol{r}_3), \qquad \boldsymbol{Q}_1 = (1/\sqrt{2})(\boldsymbol{p}_2 - \boldsymbol{p}_3),$$

or any cyclic permutation thereof related by a linear transformation. In the six-dimensional space of vectors $(\boldsymbol{X}_1, \boldsymbol{Y}_1)$ equipped with the metric $g_{\alpha\beta} = \delta_{\alpha\beta}$ one can write Schrödinger's equation and study the asymptotic properties of the scattering solution as the hyper-radius $\rho \to \infty$. The kinetic energy operator is just the Laplacian ∇^2 (from now on $\boldsymbol{\nabla}$ will denote the 6-gradient).

One can also pose (Servadio 1981b) the problem of conservation of the flux associated with the current $\boldsymbol{j} = \mathrm{Re}(\mathrm{i}\Psi^* \boldsymbol{\nabla} \Psi)$. The region around the origin corresponds to all three particles being close and interacting. Away from the origin along a ray the relative distances increase so that, if the pair potentials are of finite range, particles are eventually free. This is so provided we avoid the special directions in which two particles stay close while the third one is receding. As is well known, this is an unavoidable complication (Ginibre 1977) that has already been dealt with (Merkurev 1971, Nuttall 1971).

Very peculiar singularities are those associated with the possibility of double scattering (see figure 1) proceeding via an on-mass-shell intermediate state. From an x-space point of view it corresponds to each pair potential being translationally independent of the position of the third particle so as to appear effectively of infinite range in R_6 (see figure 2). As a consequence the (connected) double-scattering diagram falls off as $O(\rho^{-2})$ when $\rho \to \infty$, and hence more slowly than the $O(\rho^{-5/2})$ of the Green function. The distentangling of on-mass-shell from off-mass-shell contributions is a delicate process that has also been solved (Servadio 1982).

To complete the calculation of the $O(1)$ flux terms through the spherical surface of radius ρ, and hence for a final evaluation of the optical theorem (Servadio 1981b), one also needs the $O(\rho^{-3})$ terms of the wavefunction.

We shall first recall (Nuttall 1971) the wave Φ^{13} describing a collision between pair (1, 2) followed by a collision between pair (2, 3). Let $(\boldsymbol{P}, \boldsymbol{Q})$ stand for the final momenta $(\boldsymbol{P}_1, \boldsymbol{Q}_1)$, let $(\boldsymbol{P}', \boldsymbol{Q}')$ stand for the initial $(\boldsymbol{P}'_3, \boldsymbol{Q}'_3)$ and let $E = P'^2 + Q'^2$ be the energy of the incoming state. In terms of $(\boldsymbol{X}_1, \boldsymbol{Y}_1)$, which we shall denote by $(\boldsymbol{X}, \boldsymbol{Y})$, Φ^{13} is given by

$$\Phi^{13} = (2\pi)^{-9/2} 3^{-3/2} \int d^3P\, d^3Q \exp[i(\boldsymbol{P}\cdot\boldsymbol{X} + \boldsymbol{Q}\cdot\boldsymbol{Y})] \frac{T_c^{13}}{E - P^2 - Q^2 + i\varepsilon}$$

where

$$T_c^{13} = \frac{\langle \boldsymbol{Q}|t_1(E-P^2)|-3^{-1/2}(2\boldsymbol{P}'+\boldsymbol{P})\rangle\langle 3^{-1/2}(2\boldsymbol{P}+\boldsymbol{P}')|t_3(E-P'^2)|\boldsymbol{Q}'\rangle}{D(\boldsymbol{P}) + i\varepsilon}$$

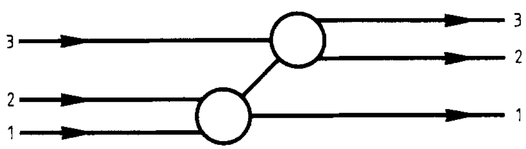

Figure 1. The double-scattering process Φ^{13}.

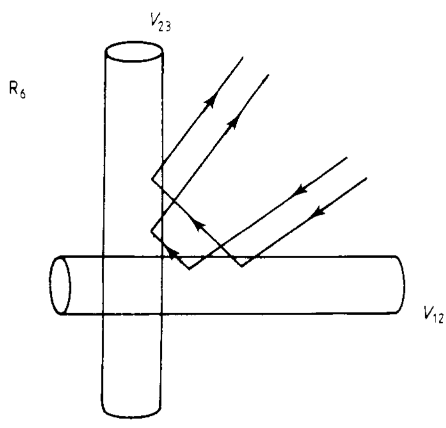

Figure 2. Two pair potentials in R_6 and parallel double-scattering 'rays'.

and

$$D(\boldsymbol{P})=\tfrac{4}{3}[\tfrac{3}{4}(E-P'^2)-(\boldsymbol{P}+\tfrac{1}{2}\boldsymbol{P}')].$$

By choosing as the new integration variable the vector $\boldsymbol{\mathscr{P}}=\boldsymbol{P}+\frac{1}{2}\boldsymbol{P}'$, in terms of which $D(\boldsymbol{P})=\frac{4}{3}(Q_0^2-\mathscr{P}^2)$ where $Q_0=\frac{1}{2}\sqrt{3}Q'$, the position of the pole is made independent of the angles of $\boldsymbol{\mathscr{P}}$ and $\boldsymbol{Q}$. This makes it possible to evaluate asymptotically as $\rho\to\infty$, at fixed $\mathscr{P}$, the inner fivefold integration in $\int \mathrm{d}\mathscr{P}\,\mathrm{d}^2\Omega_{\mathscr{P}}\,\mathrm{d}^3Q$ essentially by picking the overall energy pole contribution and adopting the stationary phase method for the four other integrations.

Since each one-dimensional stationary phase integration gives an asymptotic series with leading term $\mathrm{O}(\rho^{-1/2})$ and subsequent terms spaced by negative integer powers, the fivefold integration gives the asymptotic series

$$\mathrm{O}(\rho^{-2})+\mathrm{O}(\rho^{-3})+\mathrm{O}(\rho^{-4})+\ldots.$$

Each term carries the same phase factor

$$\exp[\mathrm{i}\rho h(\theta^*,\mathscr{P})]$$

and the intermediate state energy pole

$$(D(\boldsymbol{P})+\mathrm{i}\varepsilon)^{-1}.$$

(Refer to Servadio (1981a) for the detailed results we are making use of.)

The outermost $\int \mathrm{d}\mathscr{P}$ integration can be performed by Bleistein's method (Bleistein 1966, Bleistein and Handelsman 1975) on each term of the series exactly as on the leading term (Newton and Shtockhamer 1976). This method provides for each integral of the type

$$\mathrm{O}(\rho^{-n})\int \mathrm{d}\mathscr{P}\frac{\exp[\mathrm{i}\rho h(\theta^*,\mathscr{P})]}{Q_0^2-\mathscr{P}^2+\mathrm{i}\varepsilon}g_n(\mathscr{P})$$

its full asymptotic expansion as a sum of contributions (a) from the pole at $\mathscr{P}=Q_0$ with phase

$$\exp[\mathrm{i}\rho h(\theta^*,Q_0)],$$

(b) from the stationary point $\mathscr{P}^*$ at which

$$\mathrm{d}h(\theta^*,\mathscr{P})/\mathrm{d}\mathscr{P}=0$$

with phase

$$\exp[\mathrm{i}\rho h(\theta^*,\mathscr{P}^*)]$$

and (c) matching terms proportional to

$$F(\rho^{1/2}|h(\theta^*,\mathscr{P}^*)-h(\theta^*,Q_0)|^{1/2})\exp[\mathrm{i}\rho h(\theta^*,Q_0)]$$

where $F(\ldots)$ is Fresnel's integral defined by

$$F(t)=\int_t^\infty \exp(\mathrm{i}\tau^2)\,\mathrm{d}\tau.$$

The stationary value $\mathscr{P}^*$ depends, for given incoming kinematics, on the direction in R_6 along which $\rho\to\infty$, but the phase function $h(\theta^*,\mathscr{P}^*)$ is independent of it and equals $E^{1/2}$, so that

$$\exp[\mathrm{i}\rho h(\theta^*,\mathscr{P}^*)]=\exp(\mathrm{i}\rho E^{1/2})$$

represents a spherically outgoing wave. The pole at $\mathscr{P} = Q_0$ determines a wholly different phase profile which we shall study in great detail in § 4.

Let us recall the leading terms of the overall expansion. Introduce vectors $\bar{\boldsymbol{P}}$ and $\boldsymbol{P}_0$ defined as the values of $\boldsymbol{P} = \boldsymbol{\mathscr{P}} - \frac{1}{2}\boldsymbol{P}'$ when $\boldsymbol{\mathscr{P}}$ is, respectively, $(Q_0;\ \theta^*(Q_0), 0)$ and $(\mathscr{P}^*;\ \theta^*(\mathscr{P}^*), 0)$. The vector $\boldsymbol{P}_0$ is equal to $(\boldsymbol{X}/\rho)E^{1/2}$. The graphical construction reproduced in figure 3 translates Nuttall's system

$$(E - \bar{P}^2)^{1/2}\frac{\boldsymbol{X}}{Y} = \bar{\boldsymbol{P}} + \beta(\bar{\boldsymbol{P}} + \tfrac{1}{2}\boldsymbol{P}') \tag{2.1}$$

$$D(\bar{\boldsymbol{P}}) = 0 \tag{2.2}$$

which embodies the stationarity of the phase while keeping the intermediate state on the energy shell. Here β is a Lagrangian multiplier chosen to be positive. Define

$$G(\omega, \hat{X}) = 2^{-11/2}\pi^{-1/2}3^{-1/2}\frac{2d^2}{Q_0\sin^2\omega\{(1+\beta)[d^2(1+\beta)+P_2^2]\}^{1/2}} \tag{2.3}$$

where $d^2 = E - \bar{P}^2$ and $\omega = \tan^{-1}(Y/X)$. We can write Φ^{13} as (Servadio 1981a)

$$\begin{aligned}\Phi^{13} \sim{}& \frac{\exp[\mathrm{i}\mathscr{T}(\bar{\boldsymbol{P}})]}{\rho^2}G(\omega, \hat{X})I_{\hat{Y}}(\bar{\boldsymbol{P}})\theta(\beta) - \frac{\exp(-\mathrm{i}\pi/4)}{\pi^{1/2}}\frac{\exp[\mathrm{i}\mathscr{T}(\bar{\boldsymbol{P}})]}{\rho^2}\\ &\times G(\omega, \hat{X})I_{\hat{Y}}(\bar{\boldsymbol{P}})\,\mathrm{sgn}(\beta)F(|\mathscr{T}(\boldsymbol{P}_0) - \mathscr{T}(\bar{\boldsymbol{P}})|^{1/2})\\ &- (2\pi)^{1/2}\exp(-3\mathrm{i}\pi/4)\frac{\gamma_1\exp(\frac{1}{2}\mathrm{i}\boldsymbol{P}'\cdot\boldsymbol{X})}{\rho^{5/2}}\exp[\mathrm{i}\mathscr{T}(P_0)]\end{aligned} \tag{2.4}$$

where $\mathscr{T}(\boldsymbol{P}) = \boldsymbol{P}\cdot\boldsymbol{X} + \mathrm{d}Y$ is the phase function for which $\mathscr{T}(\bar{\boldsymbol{P}}) = \rho h(\theta^*, Q_0)$ and $\mathscr{T}(\boldsymbol{P}_0) = \rho E^{1/2}$. Moreover

$$\gamma_1 = \frac{-\mathrm{sgn}(\beta)}{[2|\mathscr{T}(\boldsymbol{P}_0) - \mathscr{T}(\bar{\boldsymbol{P}})|]^{1/2}} \times\left(\frac{f(\bar{\boldsymbol{P}})}{2} - \frac{f(\boldsymbol{P}_0)}{\mathscr{P}^* + Q_0}\frac{[2|\mathscr{T}(\boldsymbol{P}_0) - \mathscr{T}(\bar{\boldsymbol{P}})|]^{1/2}}{|\mathscr{P}^* - Q_0|}\frac{Q_0}{(\mathrm{d}^2h/\mathrm{d}\mathscr{P}^2)^{1/2}_{\mathscr{P}^*}}\right)$$

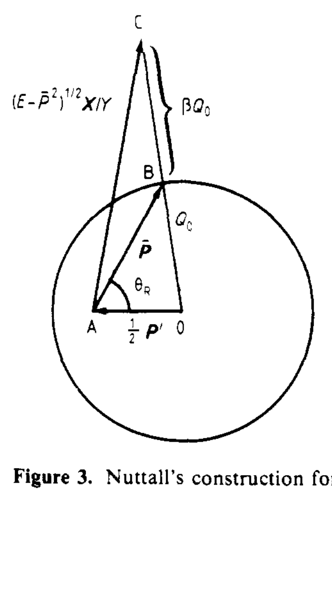

Figure 3. Nuttall's construction for equations (2.1) and (2.2).

with

$$f(\boldsymbol{P}) = G(\omega, \hat{X}) I_{\hat{Y}}(\boldsymbol{P})$$

and

$$I_{\hat{Y}}(\boldsymbol{P}) = \langle \ldots | t_1 | \ldots \rangle \langle \ldots | t_3 | \ldots \rangle.$$

The value of β depends on the direction along which $\rho \to \infty$ and $\theta(\beta) = \theta(\mathscr{P}^* - Q_0)$. In those directions where $\beta = 0$ the stationary value $\mathscr{P}^*$ coincides with the pole value Q_0; such directions represent, over the spherical surface $\rho = \text{constant}$, a four-dimensional manifold called the ridge (Newton and Shtockhamer 1976) across which there is a discontinuous asymptotic behaviour of Φ^{13}.

We shall from now on denote the non-spherical phase by $E^{1/2}\sigma = \mathscr{I}(\bar{\boldsymbol{P}})$ and write equation (2.4) in the more compact form

$$\Phi^{13} \sim \frac{\exp(\mathrm{i}E^{1/2}\sigma)}{\rho^2} A_0\left(\theta(\beta) + \operatorname{sgn}(\beta)\frac{\exp(\mathrm{i}\pi/4)}{\pi^{1/2}} F[E^{1/2}(\rho-\sigma)^{1/2}]\right)$$
$$+\frac{\exp(\mathrm{i}E^{1/2}\rho)}{\rho^{5/2}}\left[U_0 + \operatorname{sgn}(\beta) A_0 \frac{\exp(\mathrm{i}\pi/4)}{\pi^{1/2}} \frac{1}{2\mathrm{i}E^{1/2}}\left(\frac{\rho}{\rho-\sigma}\right)^{1/2}\right].$$

Firstly, we wish to point out that although the amplitude A_0 is known in terms of the momenta and of the 'direction of observation' in R_6 its actual structure, as can be seen from the awkward expression for $G(\omega, \hat{X})$, does not lend itself to any interpretation.

Secondly, away from the ridge one can substitute for F its asymptotic expansion for large arguments and cancel it against the other $\operatorname{sgn}(\beta)$ term. The resulting wave behaves as

$$\Phi^{13} \sim \frac{\exp(\mathrm{i}E^{1/2}\sigma)}{\rho^2} A_0 + \frac{\exp(\mathrm{i}E^{1/2}\rho)}{\rho^{5/2}} U_0.$$

Retracing the steps we have followed in the construction of the leading term we find that essentially the same scheme is replicated to higher orders, namely:

$$\Phi^{13} \sim \frac{\exp(\mathrm{i}E^{1/2}\sigma)}{\rho^2}\left(A_0 + \frac{A_1}{\rho} + \frac{A_2}{\rho^2} + \ldots\right)\left(\theta(\beta) + \operatorname{sgn}(\beta)\frac{\exp(\mathrm{i}\pi/4)}{\pi^{1/2}} F[E^{1/2}(\rho-\sigma)^{1/2}]\right)$$
$$+\frac{\exp(\mathrm{i}E^{1/2}\rho)}{\rho^{5/2}}\left\{\left[U_0 + \operatorname{sgn}(\beta) A_0 \frac{\exp(\mathrm{i}\pi/4)}{\pi^{1/2}} \frac{1}{2\mathrm{i}E^{1/2}}\left(\frac{\rho}{\rho-\sigma}\right)^{1/2}\right]\right.$$
$$+\frac{1}{\rho}\left[U_1 + \operatorname{sgn}(\beta) A_1 \frac{\exp(\mathrm{i}\pi/4)}{\pi^{1/2}} \frac{1}{2\mathrm{i}E^{1/2}}\left(\frac{\rho}{\rho-\sigma}\right)\right.$$
$$\left.+\operatorname{sgn}(\beta) A_0 \frac{\exp(\mathrm{i}\pi/4)}{\pi^{1/2}} \frac{1}{2\mathrm{i}E^{1/2}}\left(\frac{\rho}{\rho-\sigma}\right)^{1/2} \frac{\rho}{2\mathrm{i}E(\rho-\sigma)}\right]$$
$$+\frac{1}{\rho^2}\left[U_2 + \operatorname{sgn}(\beta) A_2 \frac{\exp(\mathrm{i}\pi/4)}{\pi^{1/2}} \frac{1}{2\mathrm{i}E^{1/2}}\left(\frac{\rho}{\rho-\sigma}\right)^{1/2}\right.$$
$$+\operatorname{sgn}(\beta) A_1 \frac{\exp(\mathrm{i}\pi/4)}{\pi^{1/2}} \frac{1}{2\mathrm{i}E^{1/2}}\left(\frac{\rho}{\rho-\sigma}\right)^{1/2} \frac{\rho}{2\mathrm{i}E(\rho-\sigma)}$$
$$\left.\left.+\operatorname{sgn}(\beta) A_0 \frac{\exp(\mathrm{i}\pi/4)}{\pi^{1/2}} \frac{1}{2\mathrm{i}E^{1/2}}\left(\frac{\rho}{\rho-\sigma}\right)^{1/2} \frac{\rho}{2\mathrm{i}E(\rho-\sigma)} \frac{1\times 3\times\rho}{2\mathrm{i}E(\rho-\sigma)}\right) + \ldots\right\}. \tag{2.5}$$

Going to infinity (away from the ridge) the $F(\ldots)$ waves cancel against the other $\mathrm{sgn}(\beta)$ terms and we are left with

$$\Phi^{13} \sim \frac{\exp(\mathrm{i}E^{1/2}\sigma)}{\rho^2}\left(A_0+\frac{A_1}{\rho}+\frac{A_2}{\rho_2}+\ldots\right)\theta(\beta)+\frac{\exp(\mathrm{i}E^{1/2}\rho)}{\rho^{5/2}}\left(U_0+\frac{U_1}{\rho}+\frac{U_2}{\rho^2}+\ldots\right). \quad (2.6)$$

This is a superposition of a wave system of phase profile $\exp(\mathrm{i}E^{1/2}\sigma)$ with a leading anomalously large rate of decrease $O(\rho^{-2})$ present only on the $\beta>0$ side of the ridge, and an everywhere present spherical system. The reader should keep in mind that the estimate does not hold where $\beta \sim 0$, so that the discontinuity is only apparent.

3. Differential equations

Any multiple-scattering wavefunction must asymptotically satisfy the Helmholtz equation if the interactions are of finite range and a direction is considered such that the pairs are asymptotically well separated. Then

$$(\nabla^2+E)\Phi^{13}=0 \quad (3.1)$$

must hold for Φ^{13} given by equation (2.5). Recalling the expansion

$$F(t) \sim \frac{\exp(\mathrm{i}t^2)}{2\mathrm{i}t}\left(1+\frac{1}{2\mathrm{i}t^2}+\frac{1\times 3}{(2\mathrm{i}t^2)^2}+\frac{1\times 3\times 5}{(2\mathrm{i}t^2)^3}+\ldots\right)$$

one can check that for Fresnel's function differentiation commutes with asymptotic expansion, that is

$$F^{(n)}(t) \sim \frac{\partial^n}{\partial t^n}\left[\frac{\exp(\mathrm{i}t^2)}{2\mathrm{i}t}\left(1+\frac{1}{2\mathrm{i}t}+\frac{1\times 3}{(2\mathrm{i}t^2)^2}+\frac{1\times 3\times 5}{(2\mathrm{i}t^2)^3}+\ldots\right)\right]$$

for any n. It follows that equation (3.1) also holds for Φ^{13} as given by equation (2.6) and, in fact, separately for the two component systems:

$$(\nabla^2+E)\frac{\exp(\mathrm{i}E^{1/2}\sigma)}{\rho^2}\left(A_0+\frac{A_1}{\rho}+\frac{A_2}{\rho^2}+\ldots\right)=0, \quad (3.2)$$

$$(\nabla^2+E)\frac{\exp(\mathrm{i}E^{1/2}\rho)}{\rho^{5/2}}\left(U_0+\frac{U_1}{\rho}+\frac{U_2}{\rho^2}+\ldots\right)=0. \quad (3.3)$$

Equation (3.3) can be dealt with by writing the Laplacian in spherical coordinates exactly as in Sommerfeld's discussion (1949) of the R_3 case; U_{n+1} is uniquely determined by U_n through a recurrence relation which is algebraic in U_{n+1}.

More complicated is equation (3.2) for the contributions from on-mass-shell intermediate states and this will be discussed in detail. Before doing so let us note that σ is a homogeneous function of degree one, i.e. $\sigma(\lambda \boldsymbol{X}, \lambda \boldsymbol{Y})=\lambda\sigma(\boldsymbol{X}, \boldsymbol{Y})$; in fact it is the phase of the Fourier transform with a definite prescription for picking poles and stationary phase contributions. Solving equation (3.2) asymptotically we find the system

$$\nabla\sigma\cdot\nabla\sigma=1, \quad (3.4)$$

$$\mathrm{i}\nabla^2\sigma\frac{A_0}{\rho^2}+2\mathrm{i}\nabla\sigma\cdot\nabla\left(\frac{A_0}{\rho^2}\right)=0, \quad (3.5)$$

$$\mathrm{i}\nabla^2\sigma\frac{A_1}{\rho^3}+2\mathrm{i}\nabla\sigma\cdot\nabla\left(\frac{A_1}{\rho^3}\right)=-E^{-1/2}\nabla^2\left(\frac{A_0}{\rho^2}\right) \quad (3.6)$$

$$\vdots \qquad\qquad \vdots$$

Equation (3.4) is the eikonal equation in free space; it states that the phase function σ has unit gradient $\hat{N} = \nabla\sigma$ everywhere.

In a recent paper (Servadio 1983) the following was proved.

Theorem. Let σ be homogeneous of degree one and of unit gradient. The inverse sets $\sigma^{-1}(s)$ are parallel surfaces; if not spherical they are $K = 0$ (ruled) surfaces.

In our case the wavefronts $\sigma^{-1}(s)$ must be ruled surfaces with at least one infinite radius of curvature.

Equation (3.5) is called the transport equation; sometimes it is called the Liouville equation. Its meaning is made clearer by recasting it into a divergence form as

$$\nabla \cdot [(A_0/\rho^2)^2 \hat{N}] = 0$$

or, by introducing the directional derivative $\nabla_{\hat{N}} = \hat{N} \cdot \nabla$, as

$$\nabla^2 \sigma = -\nabla_{\hat{N}} [\log(A_0/\rho^2)^2]. \tag{3.7}$$

The inverse sets $\sigma^{-1}(s)$ being a family of parallel surfaces, it is useful to parametrise them by surface coordinates which take the same values on all points homothetic with respect to the origin. We denote them collectively by p. Each point of R_6 is in correspondence with the pair (s, p) which will be referred to as its eikonal coordinates, s playing the role of the optical length. It is well known that the radii R_i of $\sigma^{-1}(s)$ increase linearly with coefficient one along the rays which are the (straight) integral lines of $\hat{N}$: each R_i can be written as $R_i = s + \mathcal{R}_i(p)$.

It is then easy to prove the following.

Theorem. The Laplacian of the eikonal function is equal to the trace of the Weingarten map:

$$\nabla^2 \sigma = \sum_i \frac{1}{R_i}. \tag{3.8}$$

The proof is trivial in a suitable coordinate system defined in terms of the phase profile and its directions of principal curvature (Somigliana 1919, Keller *et al* 1956).

Equations (3.7) and (3.8) together imply

$$\nabla_{\hat{N}} \left[\log \left(\frac{A_0}{\rho^2} \right)^2 \right] = \sum_i \frac{1}{R_i}.$$

Since the wavefront is $K = 0$, but otherwise arbitrary, the amplitude A_0 must have the following structure:

$$A_0/\rho_2 = g(\hat{N}) \left(\prod_i R_i^{1/2} \right)^{-1}. \tag{3.9}$$

This in turn implies the occurrence of only four finite radii of curvature.

Equation (3.6) is a first-order partial differential equation for A_1 involving only the longitudinal $\nabla_{\hat{N}}$ derivative. Essentially the same recurrence relation has often been discussed in the high-frequency asymptotics of wave mechanical problems (Keller *et* *al* 1956), in which context it is called Luneburg's recurrence relation. Its formal integration can be performed easily:

$$\frac{A_1}{\rho^3}=\frac{A_0}{\rho^2}\frac{-\mathrm{i}}{2E^{1/2}}\int_s^\infty \frac{\rho^2}{A_0}\nabla^2\left(\frac{A_0}{\rho^2}\right)\mathrm{d}r \tag{3.10}$$

where $\int_s^\infty \mathrm{d}r$ stands for integration along the ray from the point at which the left-hand side is evaluated to infinity (s is the optical length at the observation point). Once A_0 is given, the calculation of A_1 is reduced to quadratures for which it is useful to write the Laplacian in eikonal coordinates. In § 4 we shall study the geometry of the double-scattering ruled system and identify a quite natural set of eikonal coordinates.

4. Ruled wavefronts. The radii and the ridge

The ruled system is characterised by the phase factor

$$\exp(\mathrm{i}E^{1/2}\sigma)=\exp[\mathrm{i}\mathcal{T}(\bar{\boldsymbol{P}})]$$

where

$$\mathcal{T}(\boldsymbol{P})=\boldsymbol{P}\cdot\boldsymbol{X}+(E-P^2)^{1/2}Y$$

and $\bar{\boldsymbol{P}}$ is the solution of the system (2.1) and (2.2). The vector $\bar{\boldsymbol{P}}$ is a function of the point $(\boldsymbol{X},\boldsymbol{Y})$: it does not depend on $\hat{Y}$, it is the same for all points homothetic with respect to the origin, it always lies on the plane of $\boldsymbol{P}'$ and $\hat{X}$ and it is parallel to $\hat{X}$ only if $\beta=0$. The vector $\hat{N}$ is

$$\hat{N}=\nabla\mathcal{T}(\bar{\boldsymbol{P}})/|\nabla\mathcal{T}(\bar{\boldsymbol{P}})|=E^{-1/2}(\bar{\boldsymbol{P}},\mathrm{d}\hat{Y}).$$

As its integral lines are straight rays it is constant along the rays.

If $\beta=0$ it is easy to see that $\hat{N}=\rho^{-1}(\boldsymbol{X},\boldsymbol{Y})$ which is radial from the origin. The points where $\beta=0$ are those at which the ruled surface is tangent to the sphere. It follows that the $\beta=0$ point of a ruling is its point closest to the origin, the distance being s; it will be called the foot of the ruling. The set of the foot points of all the rulings of a given eikonal surface is the ridge of that surface; it is a four-dimensional manifold.

Since we know that the eikonal surfaces must be $K=0$ with the rulings as lines of vanishing curvature, $\hat{N}$ must be constant along each ruling as must the vectors $\bar{\boldsymbol{P}}$ and $\hat{Y}$. Let us call $\hat{E}_\lambda$ the unit vector field along the rulings and λ the distance from the foot (see figure 4). The vanishing of the curvature is expressed by $\nabla_{\hat{E}_\lambda}\hat{N}=0$ and $\hat{N}$ is constant over the whole two-dimensional plane generated by $\hat{N}$ and $\hat{E}_\lambda$.

Let $\boldsymbol{\rho}_\mathrm{R}$ denote vector connecting the origin to the ridge. Any position vector $\boldsymbol{p}=(\boldsymbol{X},\boldsymbol{Y})$ can be decomposed in a unique manner by first going to the appropriate foot point and then moving along the ruling according to $\boldsymbol{p}=\boldsymbol{\rho}_\mathrm{R}+\lambda\hat{E}_\lambda$ ($\lambda>0$ will correspond to $\beta>0$).

It is useful to follow Nuttall's construction in figure 3 while going down a ruling to its foot point. The point B stays fixed holding $\bar{\boldsymbol{P}}$ constant while the segment CB shrinks to zero; the vector $\boldsymbol{X}$ remains on the original plane formed with the polar axis $\boldsymbol{P}'$ (constant φ_X) while bending over (changing θ_X) so as to become parallel to $\bar{\boldsymbol{P}}$; the ratio X/Y decreases; $\hat{Y}$ stays fixed. In terms of the spherical coordinates

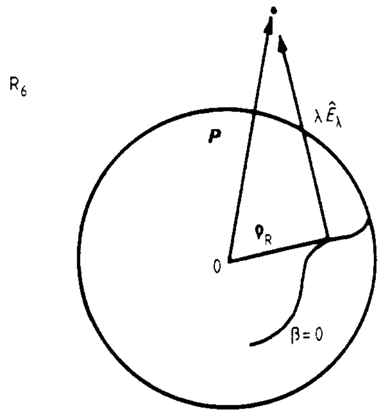

Figure 4. Eikonal coordinates. The locus $\beta=0$ is the four-dimensional ridge on the spherical surface $|\boldsymbol{\rho}_{\mathrm{R}}|=s$.

$(\rho, \omega, \theta_X, \varphi_X, \theta_Y, \varphi_Y)$ where $\omega=\tan^{-1}(Y/X)$, as $\beta \downarrow 0$ along a ruling

$$(\rho, \omega, \theta_X, \varphi_X, \theta_Y, \varphi_Y) \rightarrow (\rho_{\mathrm{R}}, \omega_{\mathrm{R}}, \theta_{\mathrm{R}}, \varphi_X, \theta_Y, \varphi_Y)$$

where the limiting values $\rho_{\mathrm{R}}, \omega_{\mathrm{R}}, \theta_{\mathrm{R}}$ have been given a suffix to remind us that they are ridge values. Recalling that the ridge of $\sigma^{-1}(s)$ lies on the sphere $\rho=s$ we can identify $\rho_{\mathrm{R}}=s$, regard ω_{R} as a function of θ_{R} and take $(\theta_{\mathrm{R}}, \varphi_X, \theta_Y, \varphi_Y)$ as the most natural parametrisation of the ridge. Ridges of different surfaces are obtained from one another by radial displacement; they are parametrised by the same $(\theta_{\mathrm{R}}, \varphi_X, \theta_Y, \varphi_Y)$.

In terms of the unit vector fields associated with the coordinates (ρ, ω, θ_X)

$$\hat{\boldsymbol{u}}_\rho=\frac{\partial \boldsymbol{p}}{\partial \rho}, \qquad \hat{\boldsymbol{u}}_\omega=\frac{\partial \boldsymbol{p}/\partial \omega}{|\partial \boldsymbol{p}/\partial \omega|}, \qquad \hat{\boldsymbol{u}}_\theta=\frac{\partial \boldsymbol{p}/\partial \theta_X}{|\partial \boldsymbol{p}/\partial \theta_X|},$$

the ruling vector field $\hat{E}_\lambda$ has the following decomposition:

$$\hat{E}_\lambda = v_\rho \hat{\boldsymbol{u}}_\rho + v_\omega \hat{\boldsymbol{u}}_\omega + v_\theta \hat{\boldsymbol{u}}_\theta.$$

It can best be computed on the ridge where v_ρ vanishes; we found

$$\hat{E}_\lambda = (-v_\omega \sin \omega_{\mathrm{R}}(\hat{X})_{\mathrm{R}} + v_\theta(\partial \hat{X}/\partial \theta_X)_{\mathrm{R}};\ v_\omega \cos \omega_{\mathrm{R}} \hat{Y})$$

where

$$v_\theta=\frac{-P_2 E^{1/2}}{|\bar{\boldsymbol{P}}|(E-P_1^2)^{1/2}}, \qquad v_\omega=\frac{-P_1 d}{|\bar{\boldsymbol{P}}|(E-P_1^2)^{1/2}},$$

$$\omega_{\mathrm{R}}=\cos^{-1}(|\bar{\boldsymbol{P}}|/E^{1/2}).$$

P_1 and P_2 are respectively the radial and the tangential components of $\bar{\boldsymbol{P}}$ in figure 3. Finally, the vector $\boldsymbol{\rho}_{\mathrm{R}}$ pointing to the ridge can be written as $\boldsymbol{\rho}_{\mathrm{R}}=(s \cos \omega_{\mathrm{R}}(\hat{X})_{\mathrm{R}}, s \sin \omega_{\mathrm{R}} \hat{Y})$. The set $(s, \lambda, \theta_{\mathrm{R}}, \varphi_X, \theta_Y, \varphi_Y)$ constitutes an orthogonal coordinate system in which λ is the length along the ruling from its foot point identified by $(\theta_{\mathrm{R}}, \omega_X, \theta_Y, \varphi_Y)$ on the eikonal surface of given s.

It is then straightforward to derive all the remaining properties of the eikonal surfaces. The coordinate vector fields have moduli

$$|\boldsymbol{E}_s| = |\boldsymbol{E}_\lambda| = 1$$

$$|\boldsymbol{E}_{\theta_R}| = \left|\frac{\partial \boldsymbol{p}}{\partial \theta_R}\right| = \frac{|\bar{\boldsymbol{P}}|^2}{P_1 d^2(E-P_1^2)^{1/2}} Y[d^2(1+\beta)+P_2^2],$$

$$|\boldsymbol{E}_{\varphi_X}| = \left|\frac{\partial \boldsymbol{p}}{\partial \varphi_X}\right| = \frac{|\bar{\boldsymbol{P}}|}{d} Y(1+\beta)\sin\theta_R,$$

$$|\boldsymbol{E}_{\theta_Y}| = \left|\frac{\partial \boldsymbol{p}}{\partial \theta_Y}\right| = Y, \qquad |\boldsymbol{E}_{\varphi_Y}| = \left|\frac{\partial \boldsymbol{p}}{\partial \varphi_Y}\right| = Y\sin\theta_Y.$$

The vectors $\boldsymbol{E}_\lambda$, $\boldsymbol{E}_{\theta_R}$, $\boldsymbol{E}_{\varphi_X}$, $\boldsymbol{E}_{\theta_Y}$ and $\boldsymbol{E}_{\varphi_Y}$ also give the directions of principal curvature. The associated radii are

$$\begin{aligned}
R_\lambda &= \infty,\\
R_{\theta_R} &= \frac{E^{1/2}}{d} Y \frac{d^2(1+\beta)+p_2^2}{d^2+p_2^2} = s + \frac{\mathrm{d}v_\theta/\mathrm{d}\theta_R - v_\omega \sin\omega_R}{\cos\omega_R}\lambda = s + c_1\lambda,\\
R_{\varphi_X} &= \frac{E^{1/2}}{d} Y(1+\beta) = s + \frac{-v_\omega \sin\omega_R \sin\theta_R + v_\theta\cos\theta_R}{\cos\omega_R \sin\theta_R}\lambda = s + c_2\lambda, \qquad (4.1)\\
R_{\theta_Y} &= \frac{E^{1/2}}{d} Y = s + \frac{v_\omega\cos\omega_R}{\sin\omega_R}\lambda = s + c_3\lambda,\\
R_{\varphi_Y} &= R_{\theta_Y}.
\end{aligned}$$

The latter representation in terms of eikonal coordinates shows the linear variation of each radius along the rays and along the rulings, as it had to be. Moreover all the radii become equal on the ridge.

The Gauss curvature K is zero since $R_\lambda = \infty$, but it is useful to consider what might be called the 'reduced' curvature

$$\mathscr{G} = \frac{1}{R_{\theta_R} R_{\varphi_X} R_{\theta_Y} R_{\varphi_Y}} = \frac{d^4}{E^2}\frac{1}{Y^4}\frac{d^2+P_2^2}{(1+\beta)[d^2(1+\beta)+P_2^2]}.$$

This quantity is related to the expansion of an infinitesimal piece of the surface as it is mapped by normal variation (Thorpe 1979) and so it must be present in the amplitude A_0. By looking at the amplitude as it was written in equations (2.3) and (2.4) one recognises that

$$\frac{A_0}{\rho^2} = \frac{G(\omega, \hat{X}) I_{\hat{Y}}(\bar{\boldsymbol{P}})}{\rho^2} = 2^{-9/2}\pi^{-1/2}3^{-1/2}\frac{E I_{\hat{Y}}(\bar{\boldsymbol{P}})}{Q_0(d^2+P_2^2)^{1/2}}\mathscr{G}^{1/2}.$$

This is in the form of equation (3.9) with

$$\mathscr{G}^{1/2} = \left(\prod_{i=1}^{4} R_i^{1/2}\right)^{-1}.$$

It affords a separation between the dynamical and the geometrical contents of the amplitude.

The reader should note that the dynamics (t-matrix elements) is contained in

$$I_{\hat{Y}}(\bar{\boldsymbol{P}}) = \langle \hat{Y} d | t_1(d^2) | -3^{-1/2}(2\boldsymbol{P}' + \bar{\boldsymbol{P}})\rangle\langle 3^{-1/2}(2\bar{\boldsymbol{P}} + \boldsymbol{P}')| t_3(Q'^2) | \boldsymbol{Q}'\rangle$$

which is only a function of $\hat{N}$. Thus, the dynamics is the same both along a ray and along each ruling. The former property was obvious to start with since a ray emerges after a well determined succession of binary scatterings. The latter property corresponds to the rays through a ruling having had the same dynamical history; they have always been parallel, so to speak (see figure 2).

Since one is ultimately interested in the dynamics, a slightly different choice is to be considered. The angle θ_R identifies $\bar{\boldsymbol{P}}$ on the circle of Nuttall's construction (at given φ_X) and consequently also the final vector $3^{-1/2}(\bar{\boldsymbol{P}}+\boldsymbol{P}')$ of $\langle\ldots|t|\ldots\rangle$, but the angle θ_i between $2\bar{\boldsymbol{P}}+\boldsymbol{P}'$ and the polar axis $\boldsymbol{P}'$ appears to be a better choice. The Jacobian is

$$\partial\theta_R/\partial\theta_i = P_1 Q_0/|\bar{\boldsymbol{P}}|^2.$$

We shall then stick to the choice

$$(s, \lambda, \xi_1, \xi_2, \xi_3, \xi_4),$$

where

$$\xi_1 = \cos\theta_i, \qquad \xi_2 = \varphi_X,$$
$$\xi_3 = \cos\theta_Y, \qquad \xi_4 = \varphi_Y,$$

play the role of ridge coordinates. The radii of curvature are obviously unchanged. The coordinate vector fields are parallel to the former ones, with moduli

$$|\boldsymbol{E}_s| = |\boldsymbol{E}_\lambda| = 1,$$
$$|\boldsymbol{E}_1| = \left|\frac{\partial \boldsymbol{p}}{\partial \xi_1}\right| = \frac{h_1(\xi_1)}{(1-\xi_1^2)^{1/2}}(s + c_1(\xi_1)\lambda),$$
$$|\boldsymbol{E}_2| = \left|\frac{\partial \boldsymbol{p}}{\partial \xi_2}\right| = (1-\xi_1^2)^{1/2} h_2(\xi_1)(s + c_2(\xi_1)\lambda),$$
$$|\boldsymbol{E}_3| = \left|\frac{\partial \boldsymbol{p}}{\partial \xi_3}\right| = \frac{h_3(\xi_1)}{(1-\xi_3^2)^{1/2}}(s + c_3(\xi_1)\lambda),$$
$$|\boldsymbol{E}_4| = \left|\frac{\partial \boldsymbol{p}}{\partial \xi_4}\right| = (1-\xi_3^2)^{1/2} h_4(\xi_1)(s + c_3(\xi_1)\lambda),$$

where

$$h_1(\xi_1) = \frac{Q_0(E-P_1^2)^{1/2}}{dE^{1/2}}, \qquad h_2(\xi_1) = \frac{Q_0}{E^{1/2}}, \qquad h_3(\xi_1) = h_4(\xi_1) = \frac{d}{E^{1/2}}$$

and $c_1(\xi_1)$, $c_2(\xi_1)$, $c_3(\xi_1)$ are the same as in (4.1). From now on, the finite principal radii of curvature will be denoted by

$$R_i = s + c_i(\xi_1)\lambda.$$

Note the functional independence of the h and R on (ξ, ξ_3, ξ_4); it stems from the rotational symmetry in $(\varphi_X, \hat{Y})$ of the eikonal surface. The volume element is

$$d^3X\, d^3Y = \sqrt{g}\, ds\, d\lambda\, d\xi_1\, d\xi_2\, d\xi_3\, d\xi_4$$

with

$$\sqrt{g} = \prod_{i=1}^{4} |\boldsymbol{E}_i| = \prod_{i=1}^{4} h_i(\xi_1)(s + c_i(\xi_1)\lambda).$$

The Laplacian is

$$\nabla^2 = \frac{1}{\sqrt{g}}\frac{\partial}{\partial s}\left(\sqrt{g}\frac{\partial}{\partial s}\right) + \frac{1}{\sqrt{g}}\frac{\partial}{\partial \lambda}\left(\sqrt{g}\frac{\partial}{\partial \lambda}\right) + \sum_{i=1}^{4}\frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi_i}\left(\frac{\sqrt{g}}{|\boldsymbol{E}_i|}\frac{\partial}{\partial \xi_i}\right). \tag{4.2}$$

These formulae provide the necessary machinery to integrate Luneburg's recurrence relation for A_1.

5. Integration along the rays

We shall from now on adopt the notation

$$A_0/\rho^2 = \mathscr{J}(\xi)\mathscr{G}^{1/2}(s, \lambda, \xi_1)$$

with

$$\mathscr{G}^{1/2}(s, \lambda, \xi_1) = \frac{1}{\Pi_i R_i^{1/2}} = \frac{1}{\Pi_i (s + c_i(\xi_1)\lambda)^{1/2}}$$

independent of (ξ_2, ξ_3, ξ_4) and

$$\mathscr{J}(\xi) = 2^{-9/2}\pi^{-1/2}3^{-1/2}\frac{EI_{\hat{Y}}(\bar{\boldsymbol{P}})}{Q_0(d^2+P_2^2)^{1/2}}$$

independent of (s, λ). The integrated recurrence relation was

$$\frac{A_1}{\rho^3} = \frac{A_0}{\rho^2}\frac{-\mathrm{i}}{2E^{1/2}}\int_s^\infty \frac{\rho^2}{A_0}\nabla^2\left(\frac{A_0}{\rho^2}\right)\mathrm{d}r. \tag{3.10}$$

The first two terms of the ∇^2 operator as given in equation (4.2) are easily dealt with since they carry $\partial/\partial s$ and $\partial/\partial\lambda$ derivatives. Also straightforward are the last three terms carrying $\partial/\partial\xi_2$, $\partial/\partial\xi_3$ and $\partial/\partial\xi_4$. We find

$$\begin{aligned}\frac{A_1}{\rho_3} = {} & \frac{A_0}{\rho^2}\frac{-\mathrm{i}}{2E^{1/2}}\left(\frac{1}{4}\sum_j\frac{1+c_j^2(\xi_1)}{R_j} - \frac{1}{2\lambda}\sum_{i<j}\frac{1+c_ic_j}{c_i-c_j}\log\frac{R_i}{R_j}\right)\\ & -\frac{\mathrm{i}}{2E^{1/2}}\mathscr{G}^{1/2}(s,\lambda,\xi_1)\int_s^\infty \mathscr{G}^{1/2}(r,\lambda,\xi_1)\frac{1}{\sqrt{g}}\\ & \times\frac{\partial}{\partial\xi_1}\left(\frac{\sqrt{g}}{|\boldsymbol{E}_1|^2}\frac{\partial}{\partial\xi_1}\right)\mathscr{G}^{1/2}(r,\lambda,\xi_1)\mathscr{J}(\xi)\,\mathrm{d}r\\ & -\frac{\mathrm{i}}{2E^{1/2}}\theta(\beta)\mathscr{G}^{1/2}(s,\lambda,\xi_1)\left[\frac{R_2}{|\boldsymbol{E}_2|^2}\frac{\partial^2}{\partial\xi_2^2}\mathscr{J}(\xi)\right.\\ & \left.+\frac{R_3}{|\boldsymbol{E}_3|^2(1-\xi_3^2)}\frac{\partial}{\partial\xi_3}\left((1-\xi_3^2)\frac{\partial}{\partial\xi_3}\right)\mathscr{J}(\xi)+\frac{R_4}{|\boldsymbol{E}_4|^2}\frac{\partial^2}{\partial\xi_4^2}\mathscr{J}(\xi)\right].\end{aligned} \tag{5.1}$$

The quadratures are elementary since the integrands are rational functions of r. Note that (5.1) contains logarithms of the radii, but many more such logarithms are obtained by integrating the $\partial/\partial\xi_1$ terms. The functions $\log(R_i/R_j)$ are homogeneous of degree zero in ρ and do not contradict the inverse power law behaviour of the wavefunction. They are, however, surprising and 'unwanted' for the following reason.

An alternative derivation of A_1 could have been carried out by improving the step-by-step asymptotic evaluation of the multiple integral outlined in § 2 for the leading term A_0. Since the stationary phase method and Bleistein's method involve higher and higher derivatives of the phase and of the integrand no such $\log(R_i/R_j)$ could ever arise. Note that $\log(R_i/R_j)$ vanishes on the ridge where the finite radii of curvature all become equal; this made us expect the occurrence of an enormous cancellation of terms.

Let us give in detail the overall coefficient of $\log R_1$ as occurring in (5.1):

$$\frac{A_1}{\rho^3}=\ldots+\frac{A_0}{\rho^2}\frac{-\mathrm{i}}{2E^{1/2}}\frac{1-\xi_1^2}{2h_1^2\lambda}$$
$$\times\Bigg[-\sum_i{}'\frac{(c_i')^2}{(c_1-c_i)^3}+c_1'\sum_i{}'\frac{c_i'}{(c_1-c_i)^3}$$
$$+\sum_{i<j}{}'\frac{c_i'c_j'(2c_1-c_i-c_j)}{(c_1-c_i)^2(c_1-c_j)^2}$$
$$-\sum_i{}'\frac{c_i''}{(c_1-c_i)^2}+\frac{2\xi_1}{1-\xi_1^2}\sum_i{}'\frac{c_i}{(c_1-c_i)^2}$$
$$+\left(\frac{h_1'}{h_1}-\sum_i{}'\frac{h_i'}{h_i}\right)\sum_i{}'\frac{c_i'}{(c_1-c_i)^2}-\frac{h_1^2}{1-\xi_1^2}\sum_i{}'\frac{1+c_1c_i}{c_1-c_i}\Bigg]\log R_1.$$

Here Σ_i' stands for $i=2,3,4$; $c_i'=\partial c_i/\partial\xi_1$, $c_i''=\partial^2 c_i/\partial\xi_1^2$ and $h_i'=\partial h_i/\partial\xi_1$. In the appendix we shall prove that the sum in the square brackets vanishes.

One can similarly spell out the overall coefficients of $\log R_i$ with $i\neq 1$ and prove that they vanish. The essential mechanism for this cancellation is exactly the same as for $i=1$. In fact the proof in the appendix makes use only of the fact that the ruled surface is immersed in a larger Euclidean (flat) space, and thus parametrised in such a way that

$$R_i=s+c_i(\xi)\lambda.$$

We have also checked that integration of the recurrence relation gives exactly the same answer as the alternative procedure of step-by-step evaluation to higher order. The fact that the two procedures yield identical results is not guaranteed on general grounds. While it is true that an asymptotic expansion has a unique assignment of coefficients once the asymptotic sequence is given (Erdelyi 1956), it is not at all clear that the multiple integral can be dealt with by successive applications of one-dimensional evaluations. To be honest, the check has here been carried out for those terms of A_1 that carry first- and second-order derivatives of two-body t-matrices. These must be considered as the most interesting from a physical point of view; their occurrence amounts to a non-trivial trace of two-body dynamics in the 'truly three-body scattering' (Servadio 1981b). However, there can be no doubt that the check extends to the full A_1 amplitude.

6. Conclusions and outlook

Let us comment on the actual usefulness of the present point of view. In a previous paper (Servadio 1981a) the $O(\rho)$ fluxes of the whole 6-current were calculated, and their cancellation was proved. Each term was proportional to the same one-dimensional integral independent of the dynamics. In view of the present paper, this integral is to be interpreted as a summation over parallel rays through the same ruling. The integration, which incidentally can be done by quadratures, admits the following appealing interpretation.

A point specified by $(\rho, \omega, \hat{X}, \hat{Y})$ corresponds, for the double-scattering wave, to well defined momenta. Retrace (see figures 2 and 5) the representative point back to make the final scattering pair (2, 3) coincident (as if with pointlike interaction). The

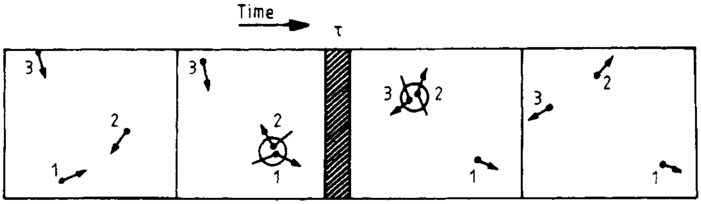

Figure 5. The intercollision time τ in Φ^{13}.

distance between the pair (1, 2) can be calculated in such a configuration and then the point can be traced back further to make (1, 2) coincide. Define the intercollision time τ as the time between the two pointlike interactions. To be explicit, τ is given by

$$\tau = \tfrac{3}{4}\rho(\beta \sin \omega)/d.$$

Consequently, the integrand of the $O(\rho)$ fluxes is none other than the exact differential $d\tau$; the integration runs from $\tau_{\min} = 0$ to $\tau_{\max} = \frac{1}{2}\sqrt{3}\rho(1/Q')$.

Conservation of the flux associated with the whole 3-3 wave Ψ is a consequence of Schrödinger's stationary equation. Since to calculate the O(1) flux terms (a program for which is about to be completed) Ψ is needed to a higher order, one would prefer to use the differential equation with as much geometric understanding as possible. If one does so, one observes many cancellations among the O(1) terms. These cancellations are parallel to those encountered in the follow-up fluxes to the optical theorem in R_3 (Servadio 1971). As in that context (Buslaev 1967), their occurrence is most naturally clarified by using the differential equation point of view.

The left out O(1) terms arising at the second-order multiple-scattering level have a very interesting structure in terms of two-body time delays and they are proportional to the same one-dimensional integral, which can be evaluated by quadratures, over parallel rays (which was also the case for the $O(\rho)$ terms, although with a different quadrature).

The occurrence of such integrations over parallel rays was to be expected for the following reason. The ultimate result for the flux calculation of the whole 6-current in x-space is a unitarity count of all the outgoing momenta. Viewed in p-space, in the CM frame and after conserving the kinetic energy, there are ∞^5 such possibilities for a three-body system. If, for example, the double-scattering sequence has been given, the dimensionality is reduced to ∞^4: one can only specify the outgoing directions of the two pair collisions. The x-space calculations then have one 'spurious' dimensionality connected to the notion of rays that are different even if parallel to one another, a distinction that cannot obtain in p-space.

This is the fundamental reason why the geometric point of view, and the detailed investigation of the actual structure, are so useful when considering the contribution from the lowest-order scattering terms and, in particular, the double-scattering processes.

Appendix. The Riemann tensor and the cancellations

Let us summarise a few facts. We have interpreted geometrically the recurrence relation for A_1 and the quadratures have generated a host of $\log(R_i/R_j)$ terms. Such logarithms are undesirable; in any case, they vanish if the radii are degenerate. We then suspect that there is a quite general geometric reason for a cancellation of such logarithms.

Look now at the overall coefficient of $\log R_1$ as in (5.2). If it vanishes, that corresponds to a differential relation on the radii of curvature of a $K=0$ surface. Such a relation must, in a sense, be universal, holding for any such surface in any dimensionality.

To be a little economical on notation, let us assume the surface to be immersed in R_4, so that there are only two finite radii of curvature (the proof in R_6 runs along the same lines). Let the element of length in R_4 be

$$(\mathrm{d}l)^2=(\mathrm{d}s)^2+(\mathrm{d}\lambda)^2+[h_1(\xi)(s+c_1(\xi)\lambda)]^2(\mathrm{d}\xi)^2+[h_2(\xi)(s+c_2(\xi)\lambda)]^2(\mathrm{d}\eta)^2$$

in terms of the differentials of the eikonal coordinates (s, λ, ξ, η). The radii are $R_i=s+c_i(\xi)\lambda$ for $i=1,2$. The vanishing of $\log R_1$ would then amount to

$$\left(-\frac{(c_2')^2}{(c_1-c_2)^3}+\frac{c_1'c_2'}{(c_1-c_2)^3}-\frac{c_2''}{(c_1-c_2)^2}+\left(\frac{h_1'}{h_1}-\frac{h_2'}{h_2}\right)\frac{c_2'}{(c_1-c_2)^2}-\frac{1+c_1c_2}{c_1-c_2}\right)=0. \tag{A1}$$

We then state the following.

Theorem. Equation (A1) holds by virtue of the flatness of the space in which the ruled surface is immersed.

Consider the Riemann tensor (Eisenhart 1949)

$$R^l{}_{ijk}=\frac{\partial}{\partial x^j}\begin{Bmatrix} l \\ ik\end{Bmatrix}-\frac{\partial}{\partial x^k}\begin{Bmatrix} l \\ ij\end{Bmatrix}+\begin{Bmatrix} m \\ ij\end{Bmatrix}\begin{Bmatrix} l \\ mj\end{Bmatrix}-\begin{Bmatrix} m \\ ij\end{Bmatrix}\begin{Bmatrix} l \\ mk\end{Bmatrix}$$

where

$$\begin{Bmatrix} l \\ ij\end{Bmatrix}=g^{ll}\frac{1}{2}\left(\frac{\partial g_{il}}{\partial x^j}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^l}\right)$$

are Christoffel's symbols in terms of the metric tensor. The Riemann tensor of R_4 is zero and this is a coordinate-independent statement. We then compute its components in the eikonal coordinate system (s, λ, ξ, η) with the diagonal metric tensor

$$g_{ss}=g_{\lambda\lambda}=1,$$
$$g_{\xi\xi}=[h_1(\xi)(s+c_1(\xi)\lambda)]^2, \qquad g_{\eta\eta}=[h_2(\xi)(s+c_2(\xi)\lambda)]^2.$$

It is an easy exercise to find (the primes here mean differentiation $\partial/\partial\xi$) that

$$R^\lambda{}_{\eta\xi\eta}=h_2 s\frac{s+c_2\lambda}{s+c_1\lambda}[(c_1-c_2)h_2'-h_2c_2']$$

and

$$\begin{aligned}R^\xi{}_{\eta\xi\eta}=\frac{h_2^2}{h_1^2}\frac{s+c_2\lambda}{(s+c_1\lambda)^2}\Bigg[&-\frac{h_2''}{h_2}(s+c_2\lambda)-2c_2'\lambda\frac{h_2'}{h_2}-c_2''\lambda\\&+\frac{h_1'}{h_1}\left(2\frac{h_2'}{h_2}(s+c_2\lambda)-\frac{c_2'}{c_1-c_2}(s+c_2\lambda)+c_2'\lambda\right)\\&+\frac{c_1'c_2'}{c_1-c_2}\lambda-h_1^2(s+c_1\lambda)(1+c_1c_2)\Bigg].\end{aligned}$$

Since $R^\lambda{}_{\eta\xi\eta}=0$, we have the equation

$$(c_1-c_2)h_2'-h_2c_2'=0 \tag{A2}$$

holding at all points; so also

$$(\partial/\partial\xi)[(c_1-c_2)h_2'-h_2c_2']=0$$

i.e.

$$\frac{h_2''}{h_2} = 2\frac{(c_2')^2}{(c_1 - c_2)^2} + \frac{c_2''}{c_1 - c_2} - \frac{c_1' c_2'}{(c_1 - c_2)^2}. \tag{A3}$$

Substituting (A2) and (A3) into the identity

$$R^{\xi}{}_{\eta\xi\eta} = 0,$$

one completes the proof of (A1).

References

Bleistein N 1966 *Commun. Pure Appl. Math.* **19** 353

Bleistein N and Handelsman R 1975 *Asymptotic Expansions of Integrals* (New York: Holt, Rinehart and Winston)

Buslaev V S 1967 *Spectral Theory and Wave Processes* ed M Sh Birman (London: Consultants' Bureau) pp 69-85

Buslaev V S and Merkurev S P 1970 *Teor. Mat. Fiz.* **5** 1216

Eisenhart L P 1949 *Riemannian Geometry* (Princeton: Princeton UP)

Erdelyi A 1956 *Asymptotic Expansions* (New York: Dover)

Faddeev L D 1965 *Mathematical Aspects of the Three-Body Problem in the Quantum Scattering* (Engl. transl. (Jerusalem: Israel Program for Scientific Translations))

Gerjuoy E 1971 *Phil. Trans. R. Soc.* **270** 197

Ginibre J 1977 *Acta Phys. Austr., Suppl.* **XVII** 95

Keller J B, Lewis R M and Seckler B D 1956 *Commun. Pure Appl. Math.* **9** 207

Kirzhnits D A and Takibaev N Zh 1978 *Sov. Phys.-JETP* **48** 397

Merkurev S P 1971 *Teor. Mat. Fiz.* **8** 235

Newton R G and Shtockhamer R 1976 *Phys. Rev.* A **14** 642

Nuttall J 1971 *J. Math. Phys.* **12** 1896

Potapov V S and Taylor J R 1977a *Phys. Rev.* A **16** 2264

—— 1977b *Phys. Rev.* A **16** 2276

Servadio S 1971 *Phys. Rev.* A **4** 1256

—— 1981a *Nuovo Cimento* B **65** 57

—— 1981b *Acta Phys. Austr., Suppl.* **XXIII** 689

—— 1982 *Nuovo Cimento* B **69** 1

—— 1983 *J. Phys. A: Math. Gen.* **16** 2997

Somigliana C 1919 *Atti Reale Acc. Sci. Torino* **54** 434

Sommerfeld A 1949 *Partial Differential Equations in Physics* (Engl. transl. (New York: Academic)) pp. 190-2

Thorpe J 1979 *Elementary Topics in Differential Geometry* (New York: Springer)